

Quasi-periodic oscillations in a network of four Rössler chaotic oscillators

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We consider a network of four non-identical chaotic Rössler oscillators. The possibility is shown of appearance of two-, three- and four-frequency invariant tori resulting from secondary quasi-periodic Hopf bifurcations and saddle-node homoclinic bifurcations of tori.

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I. INTRODUCTION

Problem of the interaction of oscillators of different nature is the focus of researchers in different fields of physics, chemistry, biology. There are examples from the domains such as optomechanical, micromechanical systems, Josephson junctions, ion traps, radio oscillators, etc. [1–8]. Individual oscillators can be periodic or chaotic. In the first case their weak interaction leads to synchronization or quasi-periodic oscillations. If you increase the number of oscillators in the system, then the number of possible incommensurate frequencies is increasing too. The result may be multi-frequency quasi-periodic oscillations, which correspond to invariant tori of higher and higher dimension [9–15]. An increase of coupling parameter may lead to breakdown of such tori with the formation of chaos. In this letter we discuss an alternative situation. We consider a network of a small number of coupled chaotic oscillators, and demonstrate the occurrence of invariant tori of different dimensions with increasing coupling parameter.

The possibility of quasi-periodic oscillations in coupled chaotic system was pointed out in [16, 17]. While investigating the ring of three unidirectionally coupled identical Lorenz systems the modes were found of not only two-, but also three-frequency quasi-periodicity. However, only one-parameter analysis was carried out and the three-frequency tori were fixed exceptionally in a very narrow range (~ 0.001) of the parameter values. The emergence of a new frequency in [16, 17] was explained by the possibility of rotational motion in the system and a high degree of symmetry due to the identity of interacting subsystems.

Here we consider the case, which is more universal. As the basic model we use a network of small number of globally coupled chaotic Rössler oscillators. The important point is the oscillators are not identical. It is well known that two such oscillators demonstrate asynchronous chaotic regimes and synchronized chaotic and periodic regimes [18–20]. In [19, 20] the "outline" of the picture of domains of different modes was presented on the "frequency detuning - coupling strength" parameter plane. Increasing the number of oscillators to four enriches and complicates this picture, which requires more detailed study. Such a task is the subject of this paper. A convenient tool for such studies is the Lyapunov analysis. Another issue we will discuss here is the question of the possible type of bifurcations of invariant tori. We have found quasiperiodic Hopf bifurcation [21], which corresponds to the soft emergence of higher dimension torus. There is also possible the saddle-node homoclinic bifurcation, typical for a resonant tori on the hypersurface of a torus of higher dimension.

II. RESULTS

Let us consider a network of four coupled Rössler oscillators

$$\begin{aligned}\dot{x}_n &= -(1 + \frac{n-1}{3}\Delta)y_n - z_n, \\ \dot{y}_n &= (1 + \frac{n-1}{3}\Delta)x_n + py_n + \frac{\mu}{3} \sum_{i=1}^4 (y_i - y_n), \\ \dot{z}_n &= q + (x_n - r_n)z_n.\end{aligned}\tag{1}$$

Here Δ parameter is introduced by analogy with [9–11] and is responsible for the frequency detuning of the oscillators. Values of the parameters $p = 0.15$, $q = 0.4$, $r = 8.5$ correspond to the chaotic regime in individual subsystems [18–20].

The system (1) may demonstrate hyperchaotic regimes with different numbers of positive Lyapunov exponents, limit cycles of different types, as well as the invariant tori of different dimensions. More detailed information about the structure of (Δ, μ) parameter plane can be obtained by constructing the Lyapunov exponents charts. In this case, at the each point in the plane the Lyapunov spectrum is computed and then the plane is painted in different colors in accordance with its signature [12–15, 21].

Lyapunov exponents chart for the system (1) is shown in Figure 1. Marked with different colors are the periodic modes P, quasiperiodic T with different number N of incommensurate frequencies, chaos C and hyperchaos CH with different number of positive exponents. In addition, there is the "amplitude death" mode AD, which is responsible for disappearance of oscillations due to their suppression by dissipative coupling. The color legend is under the figure. Numbers denote periods in the Poincare section for the simplest limit cycles.

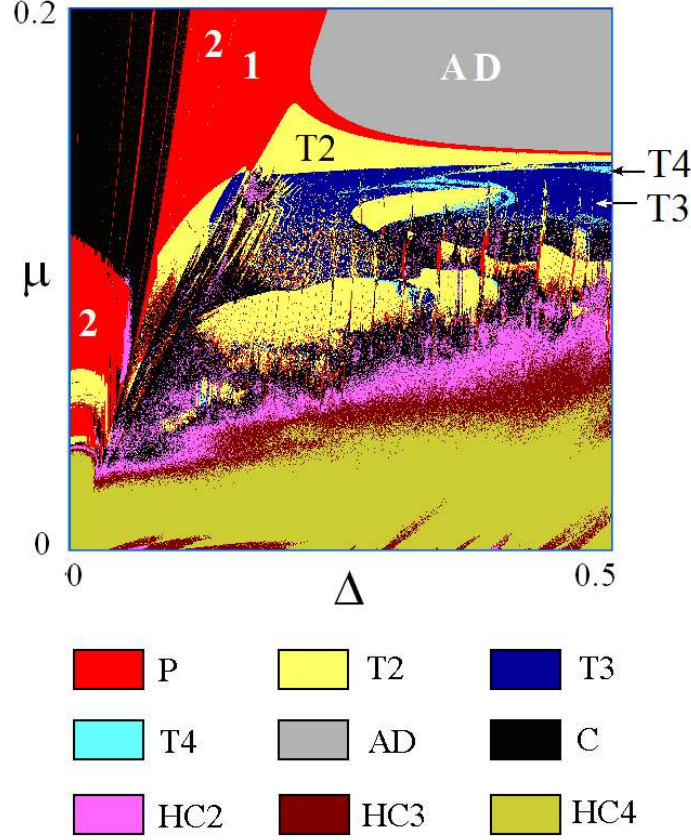


FIG. 1: Lyapunov exponents chart for the system (1), $p=0.15$, $q=0.4$, $r = 8.5$.

We first discuss the region of non-synchronous modes, which corresponds to sufficiently large frequency detuning. At low coupling the hyperchaotic mode HC4 dominates with four positive Lyapunov exponents, which is quite natural for a system of four coupled chaotic oscillators. With the increasing of coupling the number of positive exponents gradually decreases, and after the transition via HC2 and HC3 regims the usual chaos C arises. For even more stronger coupling the complicated picture of alternating of modes of different types may be observed, and typical are two-frequency T2 and three-frequency T3 tori.

Examples of invariant tori in the Poincare section at $\Delta = 0.25$ for increasing coupling parameter in the $0.098 < \mu < 0.15$ range are given in Figure 2. Poincare section was defined by conditions $y_1 = 0$, $x_1 > 0$. It can be seen that for sufficiently strong coupling a simple torus may be observed, in the Poincare section we see invariant curve close to a circle, Figure 2a. With a decrease of coupling this curve is distorted in shape. Then, three-frequency torus arises softly, Figure 2b. Next there is a sufficiently large number of various regimes transformations. For example, it can be seen on Figure 2c that the phase trajectories are condensating along some directions inside the hypersurface of a three-frequency torus. From these condensations the tangled two frequency resonant torus arises as a result of the saddle-node homoclinic bifurcation, Fig. 2d. Note that the transition from Fig.2c to Fig. 2d corresponds to a very small change in the parameter. With a further reduction of coupling another saddle-node homoclinic bifurcation of the resonant two-torus occurs, and there is again a three-frequency torus. Number of resonant windows of this type is sufficiently large.

Four-dimensional tori are also possible, however, they are less typical. They are most reliably identified at high frequency detuning. Figure 3 shows the corresponding graph for the five senior Lyapunov exponents at $\Delta=0.5$. It

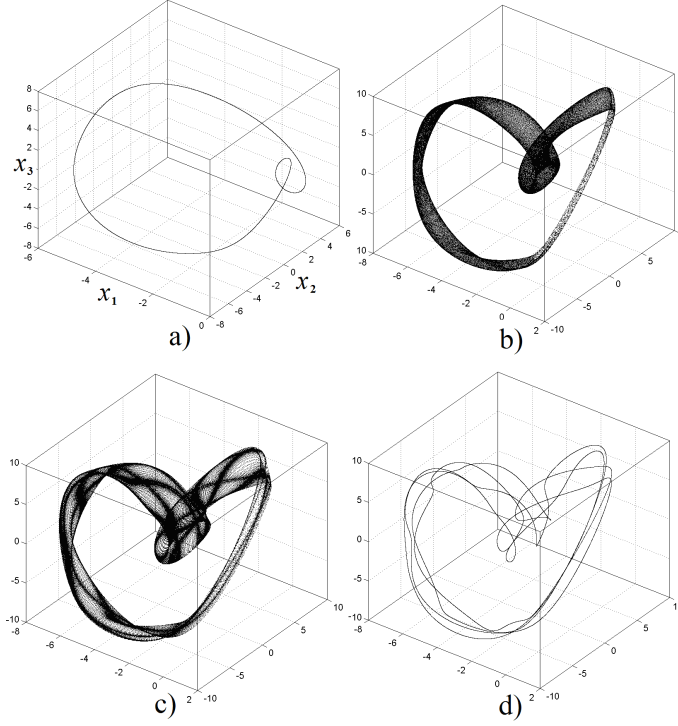


FIG. 2: Examples of Poincaré sections of two- and three-frequency invariant tori, $\Delta=0.25$; a) $\mu=0.143$, b) $\mu=0.14$, c) $\mu=0.13889$, d) $\mu=0.138919$.

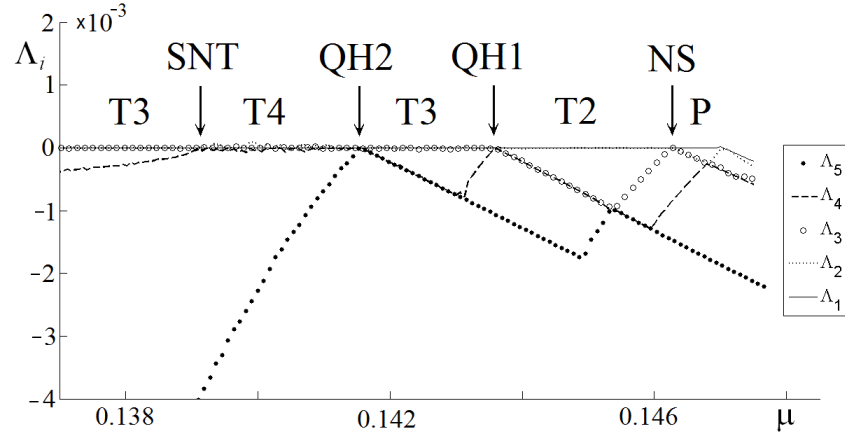


FIG. 3: Graph of five senior Lyapunov exponents vs. coupling parameter and characteristic bifurcations points, $\Delta=0.5$.

allows to describe the bifurcations observed in the system. At high coupling there is a stable equilibrium. Then from this state a stable limit cycle is born via Andronov-Hopf bifurcation (AH), this cycle at the Neimark-Sacker (NS) bifurcation point turns into a two-frequency torus. Now the first Λ_1 and the second Λ_2 exponents vanish[22]. Then at the QH1 bifurcation point the three-frequency torus is born. Note that the type of bifurcation in this case is determined by the behavior of the Lyapunov exponents. To the right of the bifurcation point the third and fourth exponents coincide, $\Lambda_3 = \Lambda_4$, and at the bifurcation point they vanish. Then, to the left, the third remains zero, while the fourth again becomes negative. This behavior is typical for a quasi-periodic Hopf bifurcation responsible for the emergence of a three-frequency torus [21]. Then, as a result of similar bifurcation QH2 the four-frequency torus is born. This time, while approaching the point of bifurcation, the fourth and fifth exponents coincide, $\Lambda_4 = \Lambda_5$. With a further decrease of coupling strength, however, there is a saddle-node homoclinic bifurcation of tori SNT, and the dimension of the torus is reduced: there is three-frequency resonant torus. In this case, the fifth Lyapunov exponent

remains negative all the time, indicating another (than QH) type of bifurcation.

Figure 4 gives a portrait of four-frequency torus in the Poincare section and the Fourier spectrum of the fourth oscillator in this case. It can be seen that the spectrum consists of individual lines, wherein there are four basic components. The remaining components correspond to the combination frequencies.

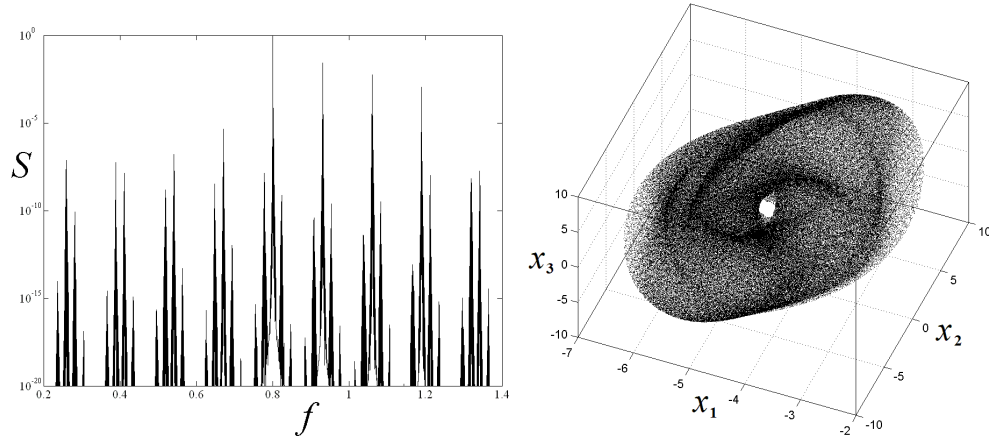


FIG. 4: Fourier spectrum and phase portrait of four-frequency torus in the Poincare section, $\Delta=0.5$, $\mu=0.14$.

In the domain of lower frequency detuning $\Delta \approx 0.3 - 0.4$ the Lyapunov chart in Figure 1 also renders four-frequency tori with four zero Lyapunov exponents. However, a more detailed analysis using charting like Figure 3 reveals also three-frequency tori with very small negative Lyapunov exponent, so the caution is required to carefully distinguish two situations.

It should be noted that our calculations show that the increase in the number of chaotic oscillators to five leads to the possibility of five-dimensional invariant tori. At the same time, however, it is advisable to also introduce non-identity of the parameter r responsible for the degree of excitation of individual oscillators.

III. CONCLUSION

Thus, for a network of chaotic oscillators for large enough frequency detuning it appears to be typical the presence of invariant tori of different dimensions, including high-dimensional ones. These tori can experience characteristic bifurcations, in particular, quasi-periodic Hopf bifurcation and saddle-node bifurcation.

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- [22] The accuracy of the numerical calculations of zero Lyapunov exponents for Figure 3 is about 10^{-5} – 10^{-6} , more accurate calculations at several chosen points demonstrate that they are zero with accuracy up to 10^{-7} – 10^{-8} .